Guided elastic waves, trapped modes and simulating scattering by fractures

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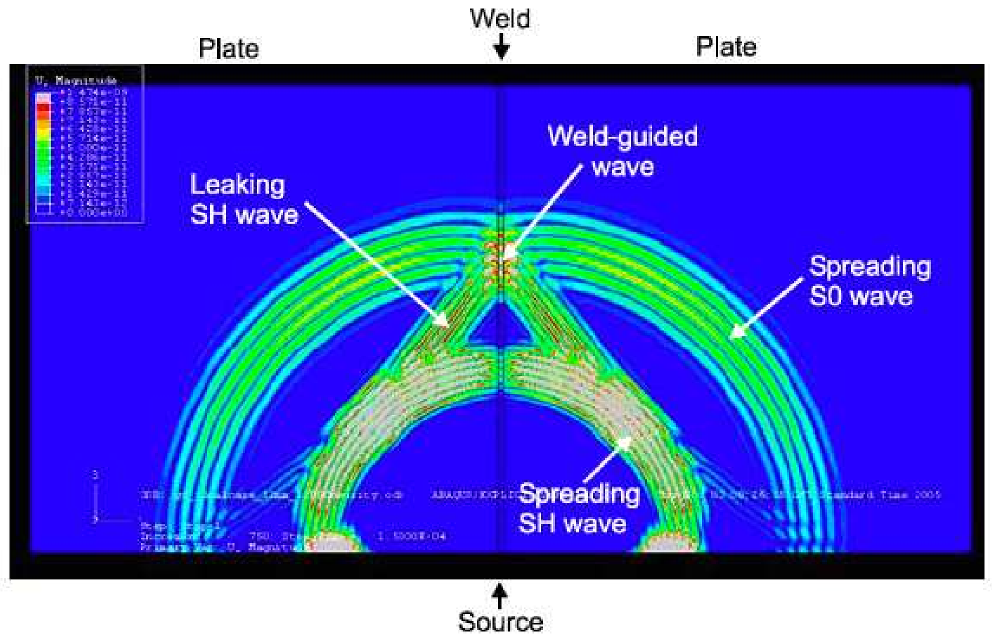
- Motivation
- Guided elastic waves
- Localised “trapped” modes
- Waves guided by topography
- Scattering and embedding
- Open issues

Many real applications in Non-Destructive Testing involve wave scattering or guiding, there are also applications in military applications - landmine detection, radar cross sections of missiles and so on.

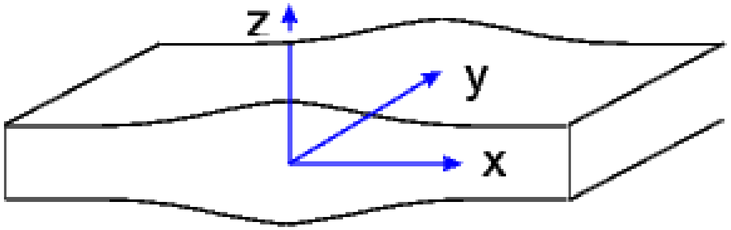
- Problem: bent waveguides can have trapped modes. These are seen in experiments and modern imaging methods are starting to use these to interogate welds (a deformed guide). At what frequencies do these occur and what is the physics? Of interest in analysis as many authors prove the existence/ uniqueness of trapped modes in semi-trivial settings, but never move to elasticity.
- Considerable practical interest in propagation along curved guides of, possibly, non-constant cross-section, curvature, width, material properties. Rock-bolts.
- Develop an asymptotic scheme to encapsulate the dominant physics. This then is an ODE and trivial to solve, versus large computations for the full problem.
- Problem: rapid evaluation of the far field of a “target” - focus on the NDT application of multiple cracks in a guided enviroment. Cracks in rail track. Typical brute force approach: Code it up in finite elements - a 20 metre rail line with 2mm cracks - tough to get accurate results. Huge computations.
- Solution, to think of an alternative to direct modelling. How about solving the “wrong” problem, but an easier one, and then manipulate the answer to replicate those of the harder physical problem - embedding.

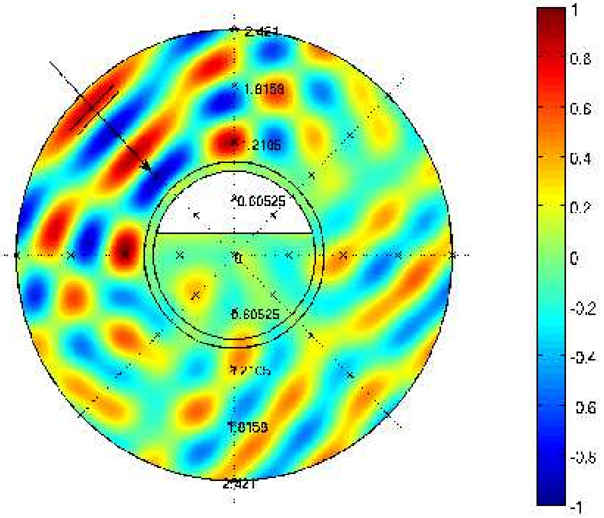
Motivation

Applications and ideas used in non-destructive testing. Guiding waves along welds using trapped modes, or along surfaces. Trying to image or interpret “hidden” quantities.



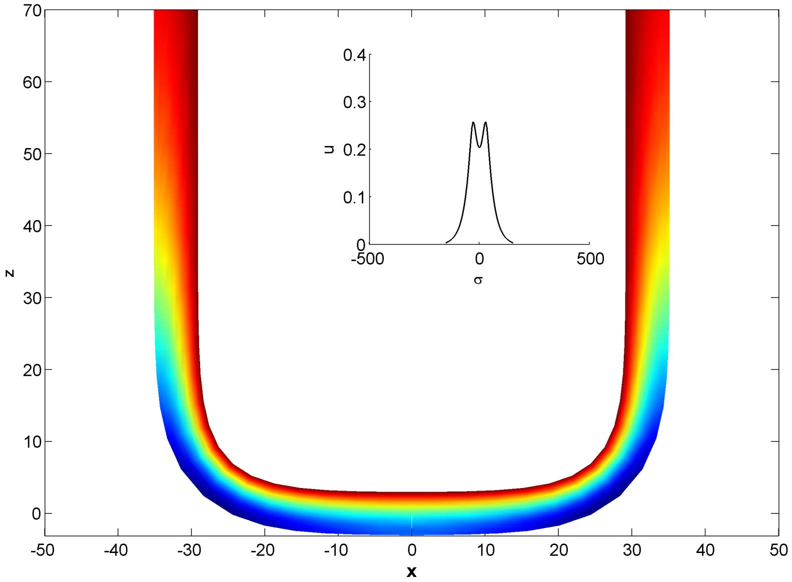
From Fan & Lowe. What are the best frequencies to use to minimise attenuation?





From Adams, Williams & Craster 2008. An incident plane wave upon a partly filled cylinder (2/3 full and steel in water). What features change as the angle of incidence change (resonances)?

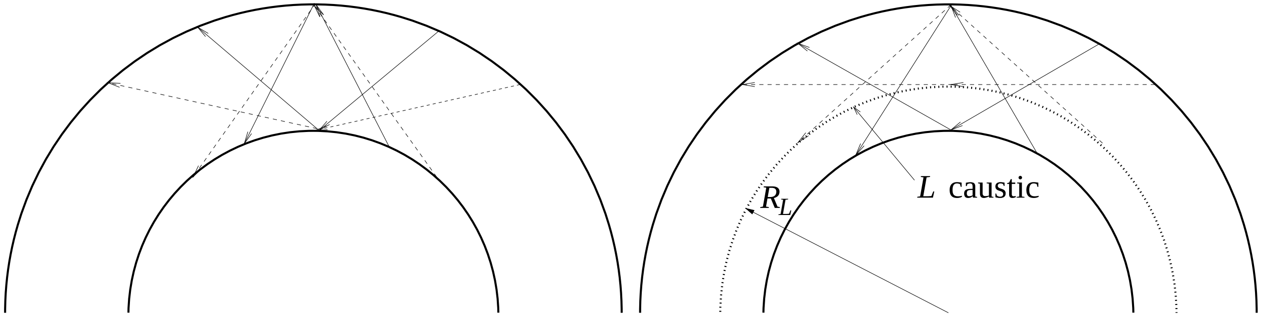
A “straight” rockbolt.

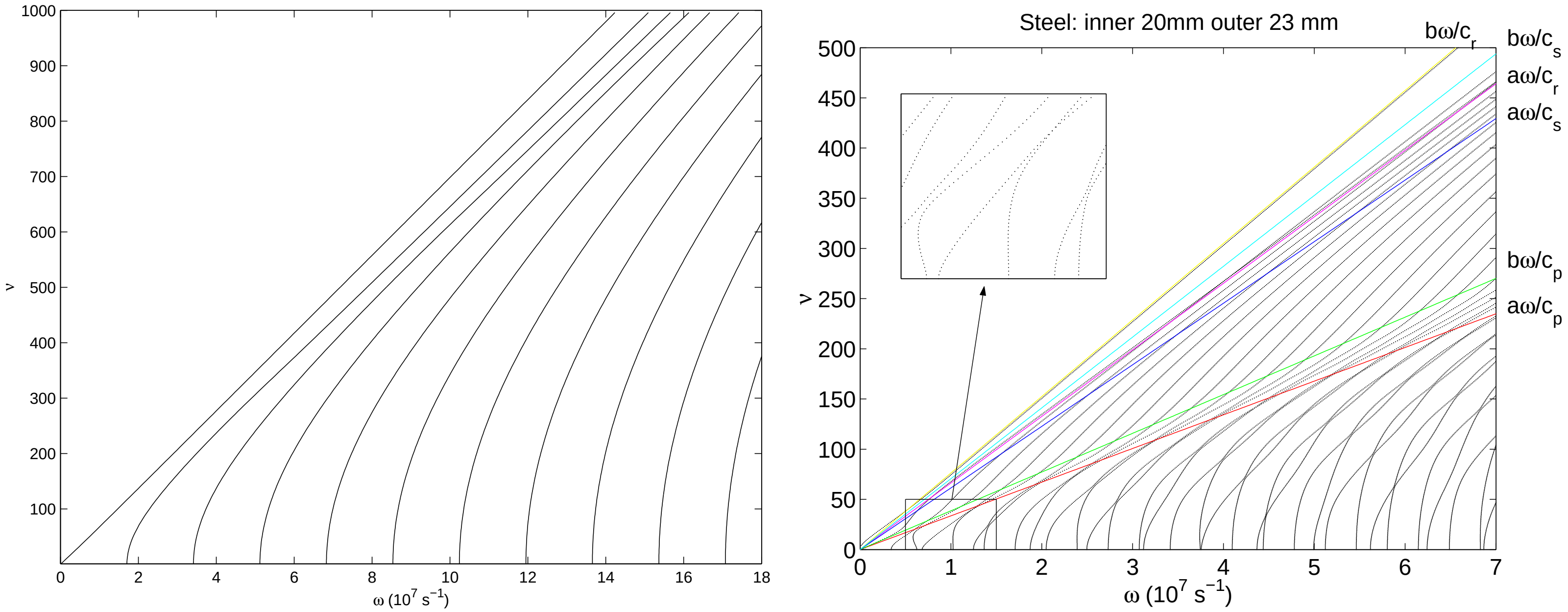


A trapped mode. It is “trapped” by the effect of curvature and occurs only for very specific frequencies. Similar effects occur if the guide gets thicker or thinner.

Dispersion curves

Typical dispersion curves relating frequency to wavenumber. Consider circumferential waves in a const curvature guide $u \sim F(r)\exp[i(\nu\theta - \omega t)]$:

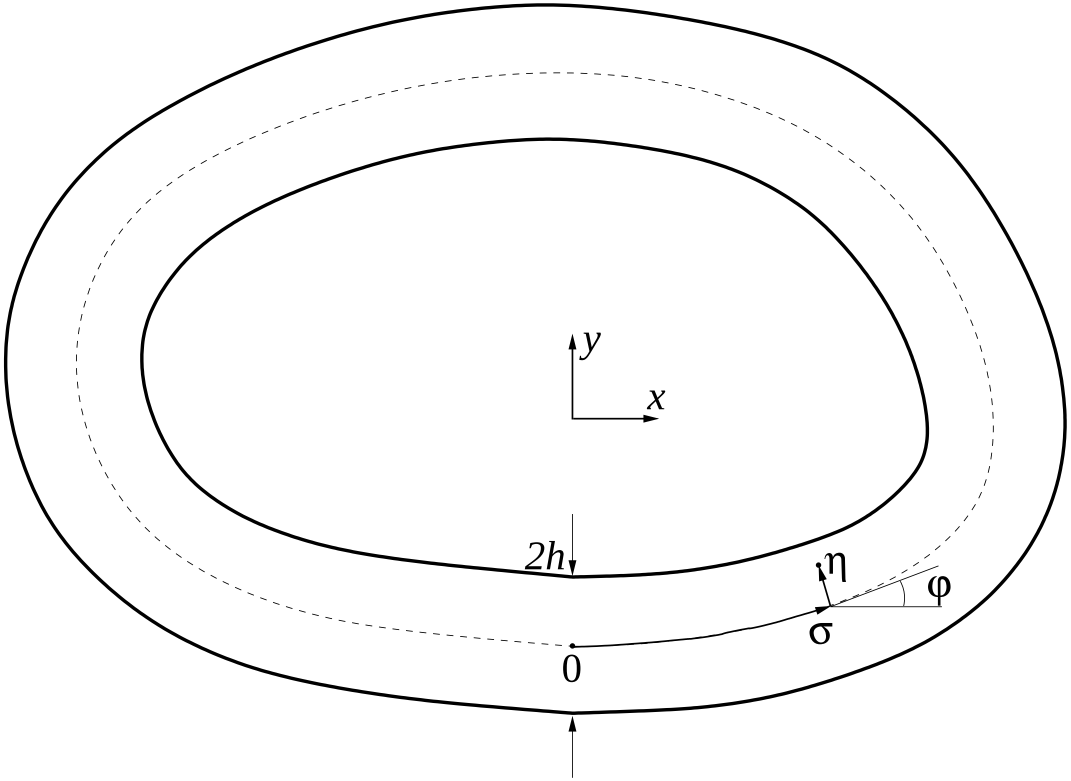




Mode “crossings”, negative group velocity, root finding. det$[D(k,\omega)] = 0$. Trivial for SH (acoustic) waves $J'_\nu(k_T a)Y'_\nu(k_T b) - J'_\nu(k_T b)Y'_\nu(k_T a) = 0$. (Neumann Bcs).

Trapping for SH waves/acoustics

Consider trapped modes and/or propagation in geometries like:



For SH waves: $u_{xx} + u_{zz} + k^2 u = 0$. In terms of σ and η it is

$$\kappa^2 u_{\sigma\sigma} + u_{\eta\eta} + \kappa^3 \varphi_{\sigma\sigma} \eta u_\sigma - \kappa \varphi_\sigma u_\eta + k^2 u = 0,$$

with $\kappa = (1 - \varphi_\sigma \eta)^{-1}$. Introduce ϵ as h/R with R typical radius of curvature.

Non-dimensionalize using h as the lengthscale. $\xi = \epsilon\sigma$, $\bar{\xi} = \xi/h$, $\bar{\eta} = \eta/h$:

$$\epsilon^2\kappa^2 u_{\bar{\xi}\bar{\xi}} + u_{\bar{\eta}\bar{\eta}} + \epsilon^3\kappa^3\varphi_{\bar{\xi}\bar{\xi}}\bar{\eta}u_{\bar{\xi}} - \epsilon\kappa\varphi_{\bar{\xi}}u_{\bar{\eta}} + \delta^{-2}u = 0,$$

where the parameter $\delta = 1/kh$ is important and $\kappa = (1 - \epsilon\varphi_{\bar{\xi}}\bar{\eta})^{-1}$.

- $kh \gg 1$ many modes are cut-on, $\delta \ll 1$.
- $kh \sim O(1)$ only a few modes are cut-on (propagate) $\delta \sim O(1)$.

In the second case can just use a modified WKBJ-ansatz:

$$u \sim \exp\left[\frac{i}{\epsilon}\sum_{k=0}^{\infty}\epsilon^k S_k(\bar{\xi},\bar{\eta})\right]\sum_{k=0}^{\infty}A_k(\bar{\xi},\bar{\eta})\epsilon^k$$

this differs from usual ansatz

$$u \sim \exp\left[\frac{i}{\epsilon}S(\bar{\xi})\right]\sum_{k=0}^{\infty}A_k(\bar{\xi},\bar{\eta})\epsilon^k$$

as the phase is now expanded, and depends on the transverse coordinate. Leading term is called adiabatic: not uniform in range. The improved ansatz leads to quasi-modes: uniform in range and satisfy the governing eqns and bcs independently. For this talk we are interested in the situation where WKBJ breaks down as $kh \to \pi m/2$.

Low n: few oscillations around guide

If there are few or no oscillations around the guide $n = 0, 1, 2$, say, we enter the regime where WKBJ fails. To proceed

$$(kh)^2 = \lambda \sim \left(\frac{\pi m}{2}\right)^2 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + ...$$

(λ now a parameter not wavelength) and

$$u(\bar{\xi}, \bar{\eta}) \sim u_0(\bar{\xi}, \bar{\eta}) + \epsilon u_1(\bar{\xi}, \bar{\eta}) + \epsilon^2 u_2(\bar{\xi}, \bar{\eta}) + ...,$$

where $u_0 = f_0(\bar{\xi}) \sin\left[\frac{\pi m}{2}(\bar{\eta} + 1)\right]$ (Dirichlet). Ulitimately one arrives at a differential eigenvalue problem for $f_0(\bar{\xi})$ and λ_2 (as $\lambda_1 = 0$) as

$$-\frac{d^2 f_0}{d\bar{\xi}^2} - \frac{1}{4}\varphi_{\bar{\xi}}^2 f_0 = \lambda_2 f_0.$$

Everything is nicely encapsulated in an ODE.
Could interpret $-\frac{1}{4}\varphi_{\bar{\xi}}^2$ attractive potential confining energy: Note for Neumann bcs the sign of that term changes and no trapped modes are found.

Eigenvalues - eigenmodes

Dimensionless eigenenergies λ_{mn} for a pseudoelliptic annulus with $a = 10$ and $b = 7$. m and n are quantum numbers, and the numbers in brackets indicate relative errors of the asymptotic and zero-curvature approximations.

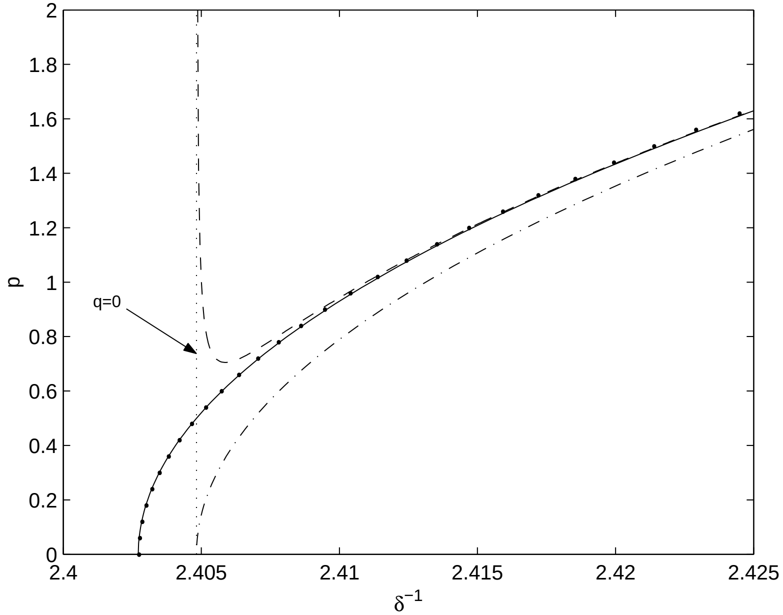
m	n	"exact"	asymptotic (rel. err.)
1	0	2.46331	2.46334 (1.2 $\times 10^{-5}$)
1	1	2.47515	2.47512 (1.3 $\times 10^{-5}$)
1	1	2.47911	2.47903 (3.3 $\times 10^{-5}$)
1	2	2.51758	2.51729 (1.2 $\times 10^{-4}$)
1	2	2.51912	2.51879 (1.3 $\times 10^{-4}$)
1	3	2.58657	2.58687 (1.2 $\times 10^{-4}$)
1	3	2.58718	2.58687 (1.2 $\times 10^{-4}$)

Physics of why trapping occurs.

Aside: one can also do a bent acoustic bar using these methods (Dirichlet bcs). Perhaps surprisingly the long wave ODE is identical in form

$$-\frac{d^2 f_0}{d\bar{\xi}^2} - \frac{1}{4}\varphi_{\bar{\xi}}^2 f_0 = \lambda_2 f_0.$$

where now $u = f_0(\xi) J_m(j_{mn}\eta)\cos m\theta$ where j_{mn} is the nth zero of $J_m(j) = 0$.



Torus: $u(\eta,\theta)\exp(\mathrm{i}p\Lambda)$ with $\Lambda = \sigma/R$. Numerical (solid), adiabatic/straight bar (dot-dash), second-order (dashed), and near-cutoff (dots) dispersion curves for the (0,1) mode close to the adiabatic cutoff frequency $1/\delta = j_{01} = 2.40483$.

Trapping in elastic guides: cut-off frequencies

For the long wave theory: There are cut-off frequencies at

$$\bar{\omega}_m^L = \frac{m\pi}{2\gamma}, \quad \bar{\omega}_m^T = \frac{m\pi}{2} \quad (m = 1, 2, 3, \dots)$$

and standard WKBJ theory fails.
Near (compressional) L cut-off symmetric (there are actually 4 cases: near L symmetric and anti-symm and near T symm and anti-symm)

$$\begin{aligned} \phi(\bar{\xi}, \bar{\eta}) \simeq \phi^{(0)}(\bar{\xi}, \bar{\eta}) + \epsilon\phi^{(1)}(\bar{\xi}, \bar{\eta}) + \epsilon^2\phi^{(2)}(\bar{\xi}, \bar{\eta}) + \dots, \\ \psi(\bar{\xi}, \bar{\eta}) \simeq \epsilon\psi^{(1)}(\bar{\xi}, \bar{\eta}) + \epsilon^2\psi^{(2)}(\bar{\xi}, \bar{\eta}) + \dots, \end{aligned} \qquad (0.1)$$

and

$$\Lambda^L \simeq (\bar{\omega}_m^L)^2 + \epsilon\Lambda_1^L + \epsilon^2\Lambda_2^L + \dots \qquad (0.2)$$

$$\phi^{(0,s)} = f^{(0)}(\bar{\xi}) \cos \frac{(2n-1)\pi\bar{\eta}}{2}, \qquad (0.3)$$

ODE

After some work, progressing through the governing equations upto ϵ^2 we get an ODE for $f^{(0)}$ as

$$C_n^{(L,s)}\frac{\mathrm{d}^2 f^{(0)}}{\mathrm{d}\bar{\xi}^2}+\left(4-\frac{1}{4\gamma^2}\right)\alpha_{\bar{\xi}}{}^2 f^{(0)}=\Lambda_2^{(L,s)}f^{(0)},$$

where we use notation

$$C_n^{(L,s)}=-\frac{16\gamma\cot[(2n-1)\pi/(2\gamma)]}{(2n-1)\pi}-\frac{1}{\gamma^2},$$

Most (all) natural materials have $\gamma>1/4$ so operator acting on $f^{(0)}$ is positive if $C_n^{(L,s)}<0$ (this means no decaying eigenfunctions at infinity and so trapping is not possible). If $C_n^{(L,s)}>0$ then a negative eigenvalue can possibly exist.
When is

$$C_n^{(L,s)}=-\frac{16\gamma\cot[(2n-1)\pi/(2\gamma)]}{(2n-1)\pi}-\frac{1}{\gamma^2}>0?$$

Negative group velocity

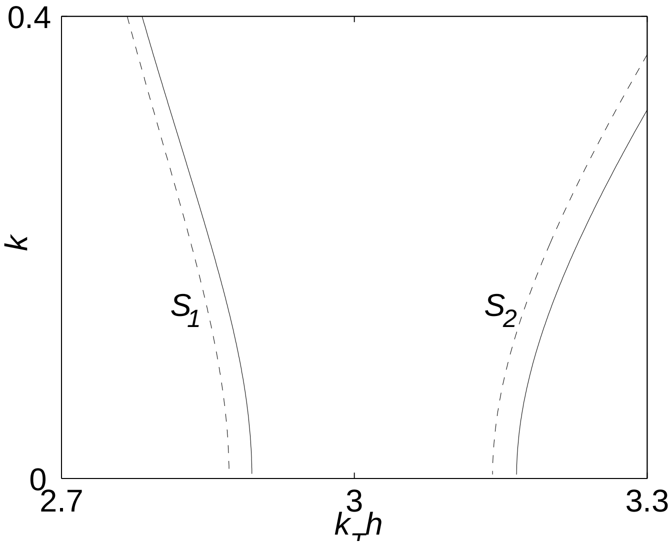
Return to a flat plate then Rayleigh-Lamb dispersion relation

$$\frac{\tan(\bar{\omega}^2 - k^2)}{\tan(\gamma^2\bar{\omega}^2 - k^2)} + \frac{4k^2\sqrt{\gamma^2\bar{\omega}^2 - k^2}\sqrt{\bar{\omega}^2 - k^2}}{(\bar{\omega}^2 - 2k^2)^2} = 0$$

has in the long-wavelength regime, $k \ll 1$,

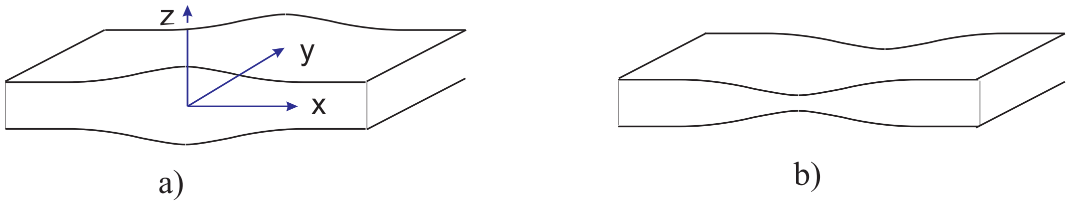
$$\bar{\omega} - \bar{\omega}^L_{2n-1} \sim -\frac{C_n^{(L,s)}}{2\bar{\omega}^L_{2n-1}}k^2. \qquad v_g = \partial\omega/\partial k \sim -\frac{C_n^{(L,s)}}{\bar{\omega}^L_{2n-1}}k$$

so $C_n^{(L,s)} > 0$ corresponds to those modes with negative group velocity (annulus solid lines).

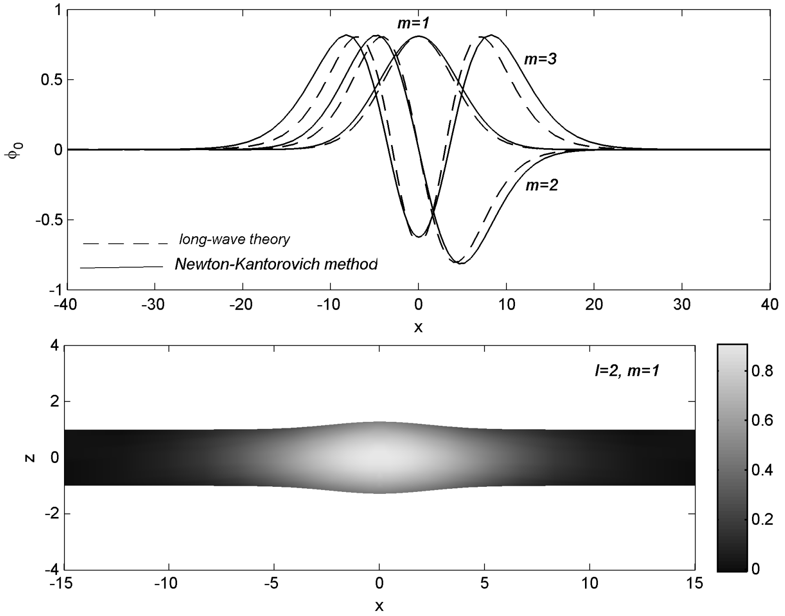


3D

Naturally one wonders whether one can guide waves along a weld: useful in NDT. This theory generates trapping within the weld and criteria again. More complicated in 3D, trapping depends on the sign of the component of the group velocity across the weld.

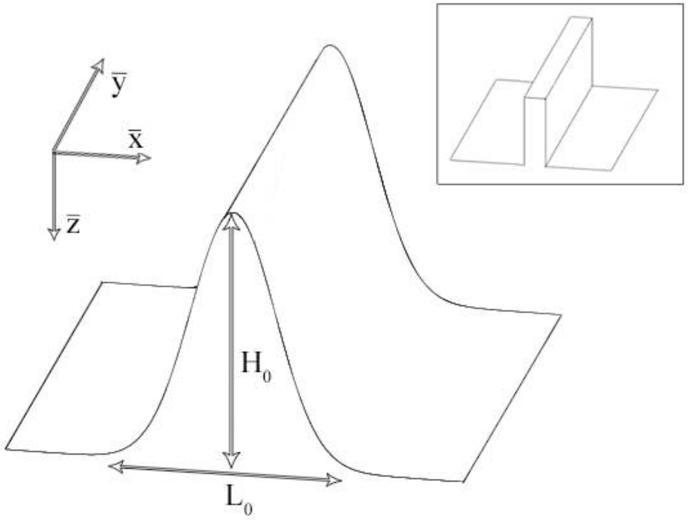


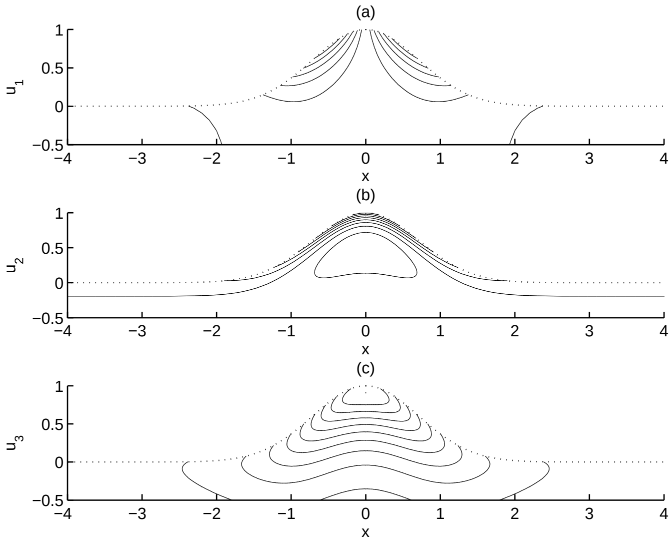
a) b)

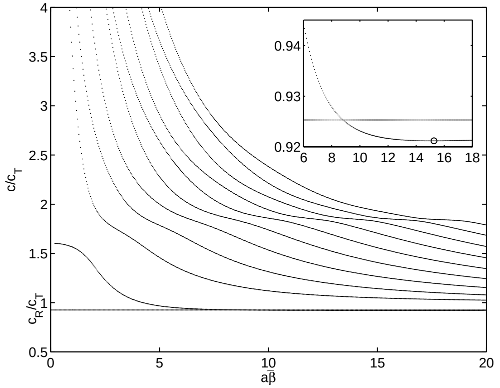


Upper panel: ϕ_0 as a function of x (for fixed $\eta = 0.8$) generated by asymptotic and direct numerical method; lower panel: ϕ_0 as a function of x and z.

Topographic “Rayleigh” waves







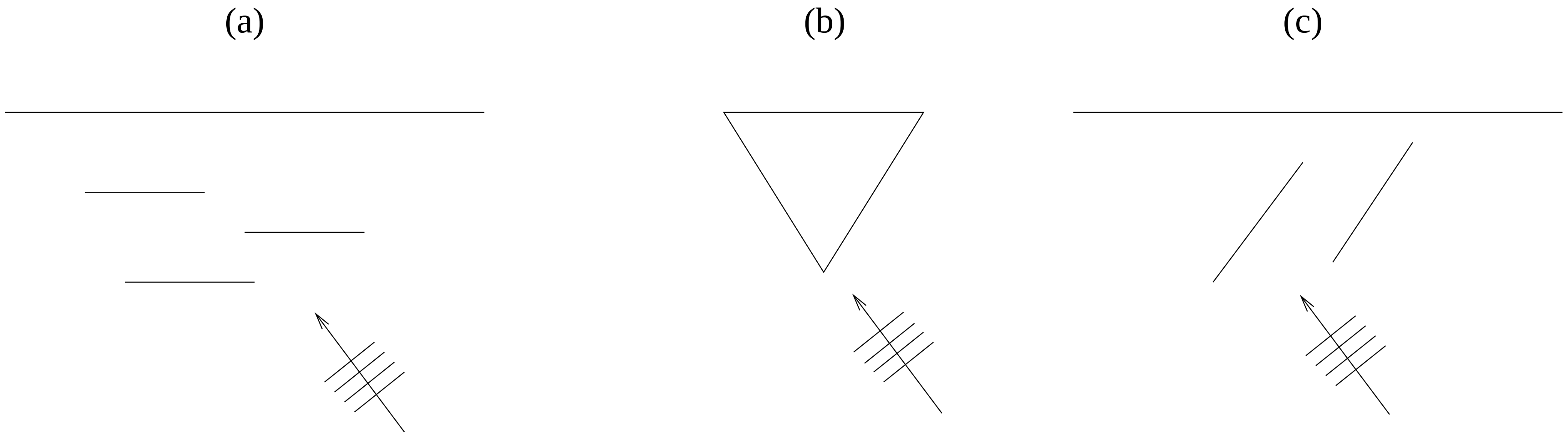
Physics is a bit different from the “simple” trapping earlier.

Summary

- WKBJ-like theories valuable for guided modes; an adjustment expanding the phase is valuable.
- Trapped modes are possible when a mode locally propagates, but is cut-off elsewhere. For elastic bent plates, high curvature allows trapping for those modes which, for the flat plate, have negative group velocities.
- If one has D-N (stress-free/ rigid) boundary conditions then trapping depends on the sign of curvature.
- Extensions to 3D and even to surface waves...

Scattering

Once the wave / guided mode etc has got to the scatterer - need to simulate the diffracted field. Each time the incoming angle/ mode changes do we have to re-run numerics. Is there a “better” way? Perhaps ...



Given

$$u^{in} = \exp\left[-\mathrm{i}k(x\cos\theta^{in} + y\sin\theta^{in})\right] = \exp\left[-\mathrm{i}kr\cos(\theta - \theta^{in})\right], \tag{0.4}$$

we often want the far field directivity $D(\theta, \theta^{in})$ for all θ, θ^{in}

$$u^{sc} \sim D(\theta, \theta^{in})\frac{\mathrm{e}^{\mathrm{i}(kR-\pi/4)}}{(2\pi kR)^{\frac{1}{2}}}, \quad \text{as} \quad R \to \infty, \tag{0.5}$$

The aim of these few overheads is to describe a fast way of doing this: instead of doing this directly, you solve a vastly reduced set of master problems and build D from them. The idea works in elasticity, in 3D, in guiding structures.

History

Embedding has a short history. Mainly from integral equation theory. Williams 1982, Porter & co-workers 1991, 2000, 2001, 2002, 2005. Reliant on formulating as an integral equation and on properties of the kernel. The word “embedding” comes from this integral equation approach. Limited by this and to the scatterers being parallel, 2D, and in acoustics.

Really you want non-parallel, 3D, elastic and implementable in any numerical scheme. The approach we develop does this. For simplicity describe in 2D & acoustics and start with parallel...

Master problem - Edge Green's function

We take as our master problem one which has multipoles at each wedge/crack tip. Natural, just like a Green's function should be...

- Deliberately chosen to be unphysically singular at the tip. For a line crack (Neumann on $x > 0$)

$$\hat{u}_j \sim r^{-j/2} \left(1 + O(r^2)\right) \cos\left(\frac{j\theta}{2}\right) \quad \text{as} \quad r \to 0. \tag{0.6}$$

  The usual near field is

$$u \sim K_1(\theta^{in}) r^{\frac{1}{2}} \cos\left(\frac{\theta}{2}\right), \tag{0.7}$$

- The master problem is also an eigensolution. It has a far field

$$\hat{u}_j \sim \hat{D}_j(\theta) \frac{\mathrm{e}^{\mathrm{i}(kR-\pi/4)}}{(2\pi kR)^{\frac{1}{2}}}, \quad \text{as} \quad R \to \infty, \tag{0.8}$$

Can we write $D(\theta, \theta^{in})$ solely in terms of $\hat{D}(\theta)$ and $\hat{D}(\theta^{in})$?

Sequence of ideas

- Using Green's theorem with the scattered field and edge Green's functions as the functions in it - or reciprocity

  $$\pi j K_j(\theta^{in}) = -2i\hat{D}_j(\theta^{in}) \tag{0.9}$$

  This relates the near field of physical problem to far field of unphysical one.
- Now we introduce a differential operator that, when applied to the physical field produces the unphysical one or a multiple of it. (The hard bit..)
- Apply uniqueness and take the far field.

The operator for parallel line cracks

Take all cracks parallel to the x axis (not necessarily on it)

$$H_1 = \frac{\partial}{\partial x} + \mathrm{i}k\cos\theta^{in}, \tag{0.10}$$

and defining a new function $\bar{u}_1$

$$\bar{u}_1 = H_1 u. \tag{0.11}$$

The operator H_1 has the following properties:

1. It maps any solution of the Helmholtz equation into another solution of the Helmholtz equation.
2. It maintains the Neumann boundary condition on the crack faces (and Dirichlet and impedance conditions if required).
3. It has the property that $H_1 u^{in} = 0$.

So what?

Near the tip

$$\bar{u}_1 = H_1 u \sim \frac{K_1(\theta^{in})}{2r^{\frac{1}{2}}} \cos\left(\frac{\theta}{2}\right) + O(r^{\frac{1}{2}}) \sim \frac{K_1(\theta^{in})}{2}\hat{u}_1 + O(r^{\frac{1}{2}}), \tag{0.12}$$

and since $H_1 u^{in} = 0$, $H_1 u^{sc} = \bar{u}_1$ too. Thus, near the tip

$$H_1 u^{sc} - \frac{K_1(\theta^{in})}{2}\hat{u}_1 = O(r^{\frac{1}{2}}) \quad \text{as} \quad r \to 0 \tag{0.13}$$

and so the left hand side of this equation satisfies the Meixner condition there. It also satisfies the Helmholtz equation, the radiation condition and the Neumann boundary condition on the crack. Thus, uniqueness applies and the left hand side is identically zero. Hence

$$H_1 u^{sc} \equiv \frac{K_1(\theta^{in})}{2}\hat{u}_1, \tag{0.14}$$

Take the far field and then for a single semi-infinite plate:

$$D(\theta, \theta^{in}) = -\frac{\hat{D}_1(\theta^{in})\hat{D}_1(\theta)}{\pi k(\cos\theta + \cos\theta^{in})}. \tag{0.15}$$

For multiple parallel cracks the same ideas go through: for N finite cracks one needs $2N$ “master” problems and then

$$D(\theta,\theta^{\rm in}) = \sum_{j=1}^{N} \frac{[\hat{D}(\theta^{\rm in};a_j^+)\hat{D}(\theta;a_j^+) - \hat{D}(\theta^{\rm in};a_j^-)\hat{D}(\theta;a_j^-)]}{\pi k[\cos\theta + \cos\theta^{\rm in}]} \tag{0.16}$$

The $a_\pm$ are the end-points of the cracks/strips. One can also do this in elasticity or for impedance bcs etc.
What if the cracks are inclined? Same ideas, but need a new operator.

Avoiding over-singularity

The idea of having to use artificially over-singular numerical solutions is off-putting - most (all?) numerical schemes usually avoid this. Do you have to re-write the entire scheme !? One can generalise this embedding idea to use a reduced set of incoming plane waves and just replace one step in the argument (Biggs 2006).

$$H_{1,\theta^{in}} u_{in}^{sc} \sim \frac{K_1(\theta^{in})}{2r^{\frac{1}{2}}} \cos\left(\frac{\theta}{2}\right), \quad H_{1,\theta_1} u_1^{sc} \sim \frac{K_1(\theta_1)}{2r^{\frac{1}{2}}} \cos\left(\frac{\theta}{2}\right).$$

For a single crack one can show that for some incoming angle θ_1:

$$H_{1,\theta_1} u_1^{sc} \equiv \frac{K_1(\theta_1)}{K_1(\theta^{in})} H_{1,\theta^{in}} u_{in}^{sc},$$

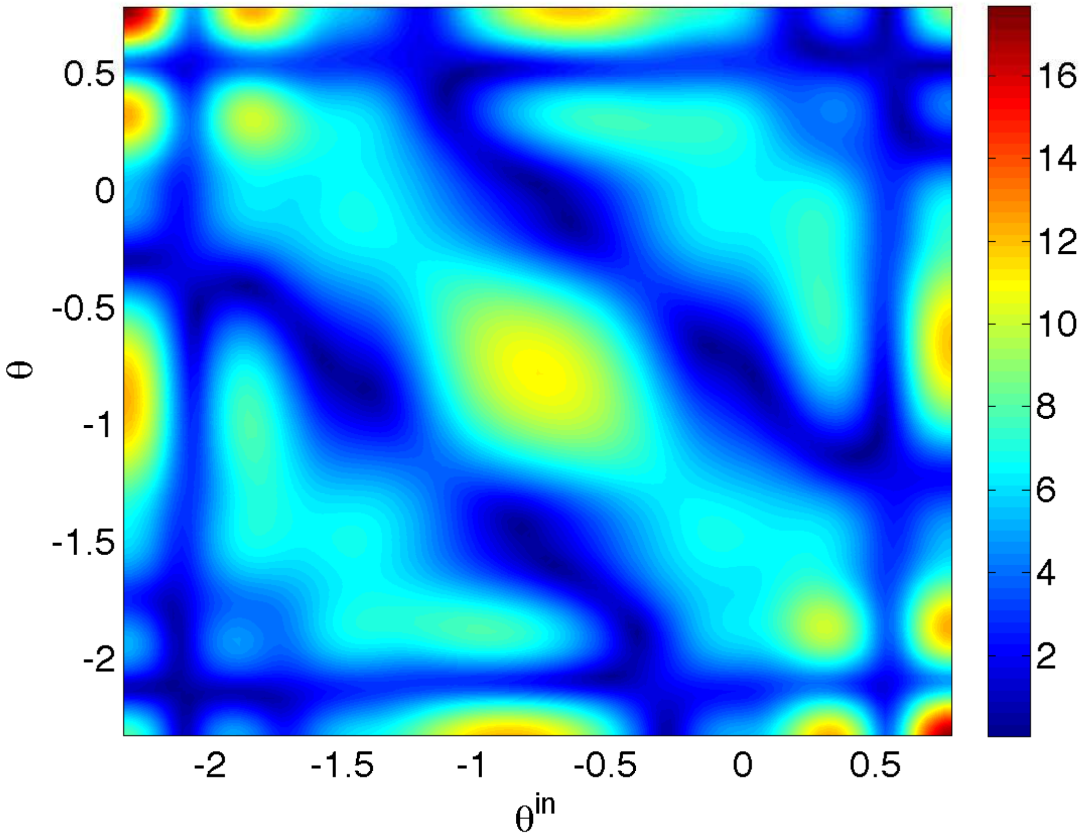
For a single semi-infinite crack given one directivity pattern for one incoming wave angle, $D(\theta, \theta_1)$ we get all others.

$$D(\theta, \theta^{in}) = \frac{(\cos\theta + \cos\theta_1)(\cos\theta_1 + \cos\theta^{in})}{2\cos\theta_1(\cos\theta + \cos\theta^{in})} \frac{D(\theta, \theta_1) D(\theta^{in}, \theta_1)}{D(\theta_1, \theta_1)}. \tag{0.17}$$

Less elegant... but in general there is a matrix formulation, N cracks and inclined etc (EAS, RVC, AVS 2007).

Numerical example

The presentation has been for parallel fractures, but at least in acoustics this can be generalised to fractures at angles to each other and to interfaces.
This extends to the other operators and for this next example (one crack, $p = 4$, need eight vertical slices to reconstruct the whole pattern). Inverse method ?
The directivity, $|D(\theta, \theta_i)|$, for a finite crack of length $2a$, $ka = 3$, whose center is a perpendicular distance $d = 2a$ from a plane inclined with $\theta_l = \pi/4$.



Summary

- For any scattering problem it should be much more efficient to use embedding. (Providing the embedding formula is known!).
- Interesting identities showing that many wavefields and solutions are inter-related. cf Babinet’s principle.
- The ideas are independent of any numerical scheme - they are more concise for overly singular master problems, but can be easily generated for plane wave incidene.
- Also works in 3D (and for surface waves, and elasticity and electro mag). Extensions to conical objects and “real” scatterers underway.

Questions for discussion

- Guiding waves efficiently to “get to” defects. Is trapping relevant to damage and fatigue? Do long-waves matter in seismology?
- Scattering and diffraction: efficient methods becoming available that would reduce calculation times. If the defect is far from the transducer are there efficient methods for crossing large distances. Similar issues in geophysics?